Patterns in the Stable sl(N) Homologies of Torus Knots

Featuring the Principle of Inclusion-Exclusion

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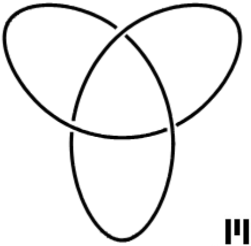
MIT PRIMES October Conference, 12 October 2024

Knots

- Continuous loops in 3-d
- Represented with 2-d diagrams



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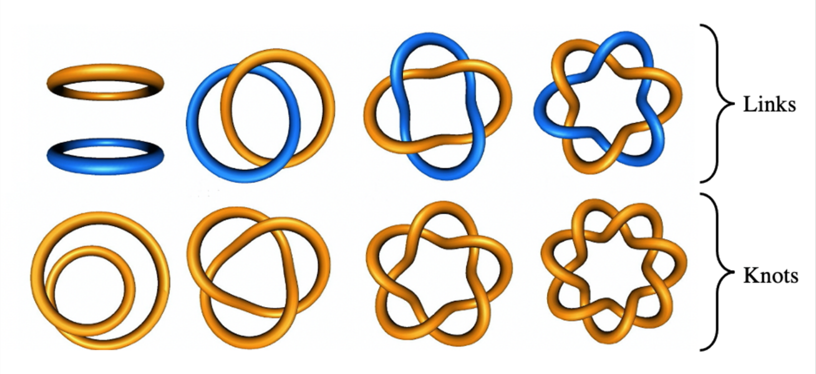
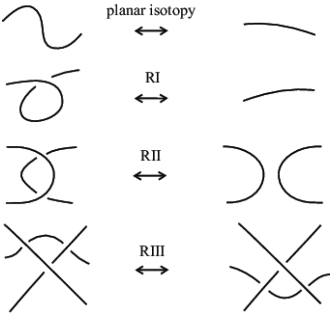
Links

- Links consist of two or more knots.



Reidemester Moves

- **Diagram Equivalence**: one can be turned into another via Reidemester moves.



The three Reidemester moves [Wei].

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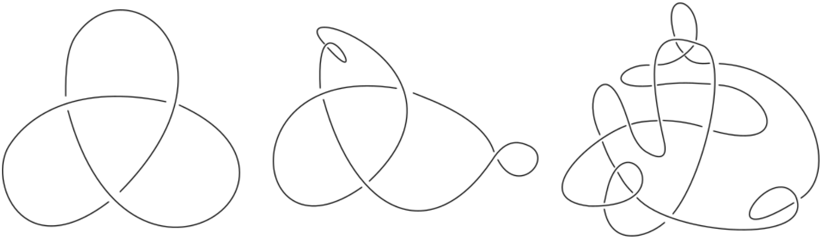
Knot Invariants

- A knot invariant I is calculated using crossings of K.
- I is a Reidemester move invariant.



Different diagrams of the trefoil [Rub+24].

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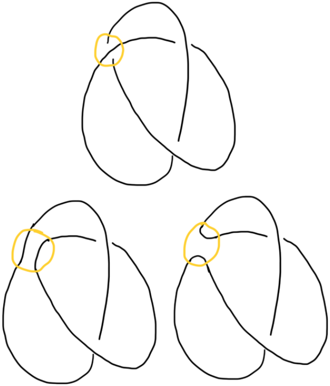
Skein Relation

- A **skein relation**: finding $I(K)$ when we know I for "similar knots."
- Depends upon the invariant.
- $I(\times) := A \cdot I()\,() + B \cdot I(\asymp)$ for some A, B (not necessarily integers or numbers).
- Base case: $I(\bigcirc)$, where $\bigcirc$ is the unknot (a circle).



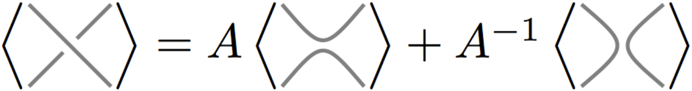
Smoothings of the trefoil.

Bracket Polynomial

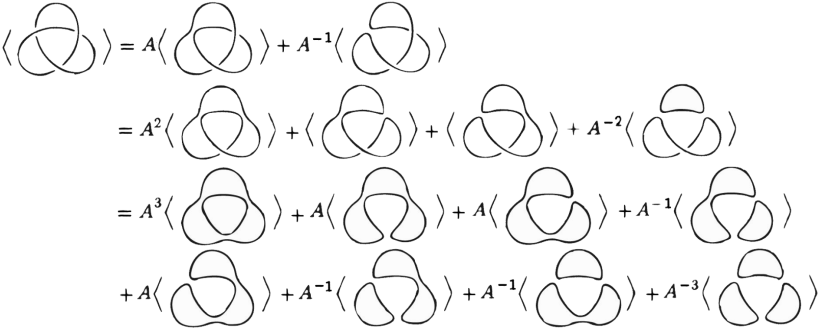
- NOT a knot invariant
- Satisfies three rules (below).
- Second rule is when we add an unknot.

1. $\left\langle \bigcirc \right\rangle = 1$

2. $\left\langle L \cup \bigcirc \right\rangle = (-A^2 - A^{-2})\langle L \rangle$

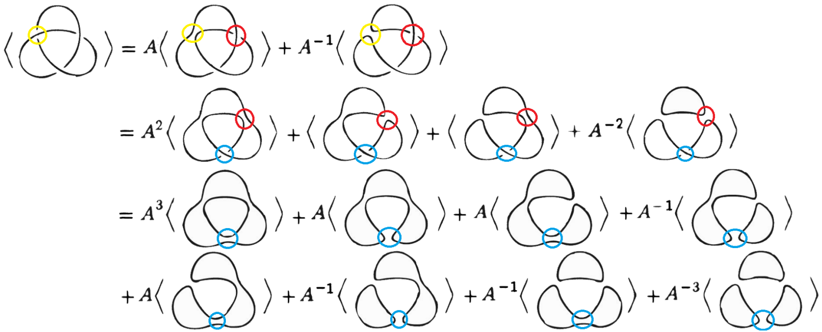
3. 

Bracket Polynomial Skein Relation Example



Bracket polynomial of trefoil [Hes].

Bracket Polynomial Skein Relation Example



Bracket Polynomial Skein Relation Example

Last expression is equivalent to...

$$A^3\left\langle \bigcirc\ \bigcirc \right\rangle + A\left\langle \bigcirc \right\rangle + A\left\langle \bigcirc \right\rangle + A^{-1}\left\langle \bigcirc\ \bigcirc \right\rangle + A\left\langle \bigcirc \right\rangle$$
$$+ A^{-1}\left\langle \bigcirc\ \bigcirc \right\rangle + A^{-1}\left\langle \bigcirc\ \bigcirc \right\rangle + A^{-3}\left\langle \bigcirc\ \bigcirc\bigcirc \right\rangle.$$

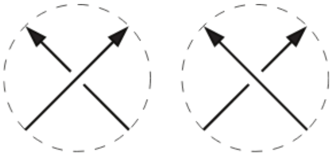
We calculate

- $\langle \bigcirc \rangle = 1$
- $\langle \bigcirc\bigcirc \rangle = -A^2 - A^{-2}$
- $\langle \bigcirc\ \bigcirc\ \bigcirc \rangle = (A^2 + A^{-2})^2 = A^4 + 2 + A^{-4}$

Substituting this into our expression, we get
$\langle$  $\rangle = A^{-7} - A^{-3} - A^5$.

Writhe and Jones Polynomial

- We assign each knot a "direction."
- The **writhe** is (# of $+$ crossings $\nwarrow\!\!\!\!\nearrow$) - (# of $-$ crossings $\nearrow\!\!\!\!\nwarrow$).
- Define $X(L) = (-A^3)^{w(L)}\langle L \rangle$.
- The **Jones Polynomial** $J(L)$ is defined by substituting $A = t^{-1/4}$ in $X(L)$.



$\epsilon(p) = +1$ $\quad\quad$ $\epsilon(p) = -1$

Khovanov homology

- A **categorification** $Kh(L)$ of the Jones Polynomial [Kho00][BN02].
- (Doubly) graded abelian group
- The Kh of the trefoil:

q/h	0	1	2	3
9				$\mathbb{Z}$
7				$\mathbb{Z}_2$
5			$\mathbb{Z}$	
3	$\mathbb{Z}$			
1	$\mathbb{Z}$			

- $J(L)$ is the **graded Euler characteristic** of $Kh(L)$.

$$J(L) = \sum_i \sum_j (-1)^j \operatorname{rank} Kh^{i,j}(L) q^i$$

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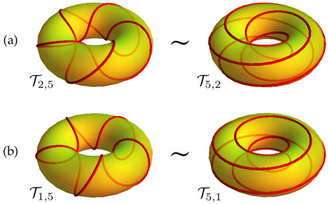
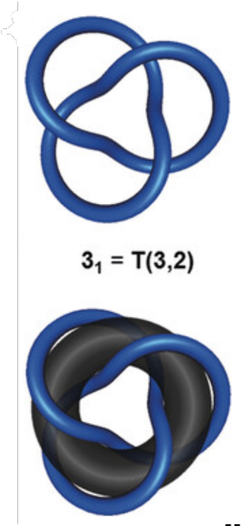
Torus Knots

- Denoted as $T(m, n)$.
- They have m "strands" which twist around the torus n times.
- Coprime m, n.



Example: Trefoil is a Torus Knot

- $T(2,3) = T(3,2) =$ the trefoil!



Stable Homology of Torus Knots

- $\lim_{n\to\infty} Kh(T(2,n))q^{-n+2}$ exists!
- The limit exists for arbitrary # of strands:

$$\lim_{n\to\infty} Kh(T(m,n))q^{-(m-1)(n-1)+1}$$

- We call these $Kh(T(m,\infty))$.
- Gorsky, Oblomkov, and Rasmussen [GOR13] conjectured $Kh(T(m,\infty)) \cong H_\bullet(W_m)$.

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3	$\mathbb{Z}$			
1	$\mathbb{Z}$			

q/h	0	1	2	3	4	5
15						$\mathbb{Z}$
13						$\mathbb{Z}_2$
11				$\mathbb{Z}$	$\mathbb{Z}$	
9				$\mathbb{Z}_2$		
7			$\mathbb{Z}$			
5	$\mathbb{Z}$					
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3	$\mathbb{Z}$					

PIE Conjecture

- If we split up those tables horizontally, we can use principal of inclusion-exclusion to compute rows further up.
- Informally, yellow = green - red.

q/h	0	1	2	3	4	5	6	7	
18									$\iddots$
16								$\mathbb{Z}$	
14								$\mathbb{Z}_2$	
12						$\mathbb{Z}$	$\mathbb{Z}$		
10						$\mathbb{Z}_2$			
8				$\mathbb{Z}$	$\mathbb{Z}$				
6				$\mathbb{Z}_2$					
4			$\mathbb{Z}$						
2	$\mathbb{Z}$								
0	$\mathbb{Z}$								

PIE Conjecture

- If we split up those tables horizontally, we can use principal of inclusion-exclusion to compute rows further up.
- Formally, where C_L is the Lth row from bottom,

$$H_\bullet(C_L) \oplus \bigoplus_{k=1}^{\lfloor \frac{n}{2} \rfloor} \bigoplus_{0<\ell_1<\ell_2<\cdots<\ell_{2k}\leq n-1} H_\bullet(C_{L-2-\sum_{i=1}^{2k} \ell_i})$$

$$\cong \bigoplus_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} \bigoplus_{0<\ell_1<\ell_2<\cdots<\ell_{2k-1}\leq n-1} H_\bullet(C_{L-2-\sum_{i=1}^{2k-1} \ell_i}).$$

Acknowledgements

I thank my PRIMES mentor, Dr. Joshua Wang, for his invaluable guidance during the reading period, impeccable expertise on the subject matter, reformulating my original conjecture into one that involves exact sequences, and providing resources and quick lessons on spectral sequences and other tools of homological algebra. I am also grateful to the PRIMES program for giving my the opportunity to work on this research project.

Citations I

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